21-241 Exam 2
Fall 2024

Name: _____

andrewID: _____

Directions: Thoroughly read the following.

Before The Exam:

- Do not open the exam until instructed to do so.
- Clearly print your name and andrewID in the designated locations above.
- Place your Carnegie Mellon ID on your desk in front of you.
- Turn off your electronic devices and store them, along with your notes and books, out of sight (e.g. in a closed bag)

During The Exam:

- Your work must be shown clearly and legibly (using a pencil or eraseable pen is always a good idea) and **you must show all work and thoroughly explain your reasoning**.
- Write your solutions only on the pages assigned to the appropriate questions. You may use the extra page(s) at the back of the exam for scratch work or continuing solutions; if you continue your solutions on the extra page(s), you should clearly indicate the page number where the solution is continued so the grader will see it. Do not remove any pages from the staples
- You may not use notes, books, other reference materials, or electronic devices.
- If you need to leave the classroom (e.g. to use the restroom), please raise your hand, turn in your CMU ID card to a proctor, and leave your electronic devices in the classroom.

After The Exam

- Stop working immediately when you are instructed to do so.
- You must hand in your own exam and you must hand in all pages of your exam.

1. Note that this problem spans two pages. Determine whether the following statements are true or false. If a statement is true, prove it; if it is false, provide a counterexample.

   (a) Every Markov matrix has a unique steady state vector.

   (b) If A is an $n \times n$ diagonalizable matrix having an eigenvalue of $\lambda = 1$ with algebraic multiplicity n, then $A = I$.

(c) If A is an $n \times n$ matrix with orthogonal columns, then $||A\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$.

(d) Every orthogonally diagonalizable matrix is symmetric.

2. Note that this problem spans two pages. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

   (a) Use the Gram-Schmidt process to find an orthonormal basis of W.

(b) Explain how the four fundamental subspaces can be used to find a basis for the orthogonal complement of W (no computation required).

(c) Determine the projection of $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ onto $W^{\perp}$.

[Hint: there is a quick way to do this and a long way to do this]

3. Note that this problem spans two pages. Let A be a 4×4 matrix with QR factorization $A = QR$, where $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

   (a) Determine the possible values of the determinant of A.

   (b) Compute A^TA.

(c) Is $A^T A$ positive definite?

(d) What does the Spectral Theorem tell you about $A^T A$?

4. Note that this problem spans three pages. Let $A = \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & 1 \end{bmatrix}$.

   (a) Determine the eigenvalues of A.

   (b) Find a basis for each eigenspace of A.

(c) Diagonalize A.

(d) Let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Find a formula for $A^k\mathbf{x}$ as a single vector.

(e) Let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Show that $A^k\mathbf{x}$ approaches one of the eigenspaces as $k \to \infty$. Which eigenspace is it?

This page is for scratch work or continued solutions